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An integrable Poisson map generated from the eigenvalue problem of the Lotka–Volterra hierarchy

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Abstract
A 3×3 discrete eigenvalue problem associated with the Lotka–Volterra hierarchy is studied and the corresponding nonlinearized one, an integrable Poisson map with a Lie–Poisson structure, is also presented. Moreover, a 2×2 nonlinearized eigenvalue problem, which also begets the Lotka–Volterra hierarchy, is proved to be a reduction of the Poisson map on the leaves of the symplectic foliation.

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1. Introduction

A symplectic map which preserves the symplectic forms is referred to as a discrete-time Hamiltonian system. Based on a discrete version of the Arnold–Liouville theorem, Veselov introduced the definition of an integrable symplectic map [1]: a symplectic map on a $2n$-dimensional symplectic manifold is said to be integrable if it admits n independent integrals in involution. Veselov's paper, together with [2,3], constructed the theory of the discrete version of classical integrable systems. Since then the investigation of this field has made important progress. Some methods have been developed to obtain new integrable symplectic maps [4–6], among which the nonlinearization technique [7–11] or the restricted flow technique [12] have proved to be effective.

The integrable maps in the above literature were based on the symplectic manifolds whose Poisson structure is nondegenerate. In [13–17], it has been shown that the Lie–Poisson structure associated with a Lie algebra is a generalized Hamiltonian structure on a Poisson manifold. The aim of the present paper is to construct the discrete version of the generalized finite-dimensional integrable Hamiltonian system, that is, integrable Poisson maps [18], on the

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Poisson manifold. A Poisson map $\varphi : M^{3N} \longrightarrow M^{3N}$ is said to be completely integrable on the Poisson manifold M^{3N} if it has N independent Casimir functions and N independent conserved integrals in involution. In this paper, a discrete 3×3 eigenvalue problem is introduced with the help of a Lie group homomorphism. It is shown that the 3×3 eigenvalue problem has the same isospectral evolution equations [19] as the 2×2 one. A constraint between the potentials and eigenfunctions is proposed, from which the 3×3 eigenvalue problem is nonlinearized to be a completely integrable Poisson map with a Lie–Poisson structure on the Poisson manifold R^{3N}. As a reduction of the integrable Poisson map on the co-adjoint orbit, an integrable symplectic map on the symplectic manifold R^{2N} is obtained, which is exactly the nonlinearized 2×2 eigenvalue problem. There is a systematic method to find constraints by symmetry constraints [20, 21]. Here we construct the required constraint with the help of the stationary discrete zero-curvature equations, which yield the Lax representation and conserved integrals in a natural way.

The organization of this paper is as follows. In section 2, the Lie group homomorphism from $SL(2, R)$ to some subgroup of $SL(3, R)$ is introduced and the corresponding Lie algebra isomorphism as the tangent map is also presented. These maps provide the basic matrix elements to study the spectral problems. In section 3, a discrete 3×3 matrix spectral problem, which has the same isospectral evolution equation with the 2×2 matrix spectral problem, is proposed. Moreover, the relation between the two spectral problems is discussed. In section 4, we give the commutative representation of the Lotka–Volterra vectors by the 3×3 spectral problem. In section 5, under a constraint between the potentials and eigenfunctions, we obtain a Poisson map, which is the nonlinearization of the 3×3 matrix spectral problem. Moreover, the Lie–Poisson structure and the integrability of this Poisson map are studied. The paper closes with section 6 where the integrable symplectic map is presented, which is exactly a reduction of the Poisson map on the co-adjoint orbit.

It is also worth noting that important work on the construction of the 2×2 and 3×3 matrix eigenvalue problems for the same hierarchy of AKNS systems is given in [22], which deals with the finte-dimensional integrable system with the help of binary nonlinearization theory.

2. Preliminaries

Similar to the framework [18], we define a map

$$\alpha \longmapsto \mathcal{L}_\zeta(\alpha) = \begin{pmatrix} \alpha_1 & \zeta\alpha_2 \\ \zeta\alpha_3 & -\alpha_1 \end{pmatrix} \tag{2.1}$$

which is a Lie algebra isomorphism between R^3 with Lie bracket $[\alpha, \beta] = C_\lambda(\alpha \times \beta)$ and the $sl(2, R)$, where

$$C_\lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \tag{2.2}$$

Define $\pi : SL(2, R) \to SL(3, R)$ by

$$\mathcal{L}_\lambda(\pi_\lambda(g)\alpha) = \mathrm{Ad}_g\mathcal{L}_\lambda(\alpha) = g\mathcal{L}_\lambda(\alpha)g^{-1}; \tag{2.3a}$$

that is,

$$\pi : g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \pi(g) = \begin{pmatrix} ad + bc & -\zeta ac & \zeta bd \\ -2ab/\zeta & a^2 & -b^2 \\ 2cd/\zeta & -c^2 & d^2 \end{pmatrix}, \tag{2.3b}$$

and we have

$$\pi(SL(2, R), \zeta) = \{\pi(g) : (\pi(g))^T C_\lambda^{-1}\pi(g) = C_\lambda^{-1}, g \in SL(2, R)\}. \tag{2.4}$$

Consider the tangent map of π, $T_e\pi : sl(2, R) \longrightarrow sl(3, R)$:

$$T_e\pi : A = \mathcal{L}_\zeta(\alpha) \longmapsto T_e\pi(A) = \begin{pmatrix} 0 & -\lambda\alpha_3 & \lambda\alpha_2 \\ -2\alpha_2 & 2\alpha_1 & 0 \\ 2\alpha_3 & 0 & -2\alpha_1 \end{pmatrix}, \tag{2.5}$$

which yields the Lie algebra of the Lie group $\pi(SL(2, R), \zeta)$:

$$T_e\pi(sl(2, R), \zeta) = \{T_e\pi(A) : (T_e\pi(A))^T C_\lambda^{-1} + C_\lambda^{-1} T_e\pi(A) = 0, A \in sl(2, R)\}. \tag{2.6}$$

By direct calculations we find that the map

$$\sigma_\lambda : \alpha \longmapsto \sigma_\lambda(\alpha) = T_e\pi(\mathcal{L}_\zeta(\alpha)) \tag{2.7}$$

satisfies

$$[\sigma_\lambda(\alpha), \sigma_\lambda(\beta)] = \sigma_\lambda(\sigma_\lambda(\alpha)\beta) = \sigma_\lambda(C_\lambda(\alpha \times \beta)) \tag{2.8}$$

and

$$\sigma_\lambda(\pi(g)\alpha) = \mathrm{Ad}_{\pi(g)}\sigma_\lambda(\alpha). \tag{2.9}$$

3. Two discrete eigenvalue problems

Let E be the shift operator: $Ef(n) = f(n+1)$, $E^{-1}f(n) = f(n-1)$. Consider the following two discrete eigenvalue problems [23]:

$$E\psi = g\psi, \tag{3.1}$$

$$Ey = \pi(g)y \tag{3.2}$$

with

$$g = \begin{pmatrix} 0 & 1/\sqrt{c_n} \\ -\sqrt{c_n} & \zeta/\sqrt{c_n} \end{pmatrix}, \qquad \pi(g) = \begin{pmatrix} -1 & 0 & \lambda/c_n \\ 0 & 0 & -1/c_n \\ -2 & -c_n & \lambda/c_n \end{pmatrix}. \tag{3.3}$$

Define two maps: $\mathcal{L}_g : R^3 \longrightarrow gl(2, R)$ by

$$\mathcal{L}_g(\gamma) = \mathcal{L}_\zeta(\gamma)g \tag{3.4}$$

and $\sigma_{\pi(g)} : R^3 \longrightarrow gl(3, R)$ by

$$\sigma_{\pi(g)}(\gamma) = \sigma_\lambda(\gamma)\pi(g). \tag{3.5}$$

One can easily check that $\mathcal{L}_g$, $\sigma_{\pi(g)}$ are linear one to one maps. Set $V = \mathcal{L}_\zeta(DG)$, $\tilde{V} = \sigma_\lambda(DG)$, with

$$D = \begin{pmatrix} \frac{1}{2}[(E^{-1}c - \lambda)(1 + E^{-1}) - c(1 + E)] \\ 1 + E^{-1} \\ -c(1 + E) \end{pmatrix} c,$$

then

$$(EV)g - gV = \mathcal{L}_g[E(DG) - \pi(g)(DG)] = \mathcal{L}_g\left[T\begin{pmatrix} (K - \lambda J)G \\ 0 \\ 0 \end{pmatrix}\right], \tag{3.6}$$

$$(E\tilde{V})\pi(g) - \pi(g)\tilde{V} = \sigma_{\pi(g)}[E(DG) - \pi(g)(DG)] = \sigma_{\pi(g)}\left[T\begin{pmatrix} (K - \lambda J)G \\ 0 \\ 0 \end{pmatrix}\right], \tag{3.7}$$

where

$$T = \begin{pmatrix} -1/2c & 0 & 0 \\ 0 & 1 & 0 \\ -1/c & 0 & 1 \end{pmatrix} \tag{3.8}$$

and

$$K = c[(1+E)c(1+E) - (1+E^{-1})c(1+E^{-1})]c, \qquad J = c(E - E^{-1})c \tag{3.9}$$

are the Lenard pair of operators.

Proposition 1. *Equations (3.1) and (3.2) have the same isospectral evolution equation:*

$$c_t = (K - \lambda J)G. \tag{3.10}$$

Proof. Let

$$\psi_t = V\psi, \tag{3.11}$$

$$y_t = \tilde{V}y, \tag{3.12}$$

then the compatibility conditions of Lax pairs (3.1), (3.11) and (3.2), (3.12) lead to the discrete zero-curvature equations:

$$g_t - (EV)g + gV = \mathcal{L}_g\left\{T\left[\begin{pmatrix} c_t \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} (K-\lambda J)G \\ 0 \\ 0 \end{pmatrix}\right]\right\} = 0, \tag{3.13}$$

$$(\pi(g))_t - (E\tilde{V})\pi(g) + \pi(g)\tilde{V} = \sigma_{\pi(g)}\left\{T\left[\begin{pmatrix} c_t \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} (K-\lambda J)G \\ 0 \\ 0 \end{pmatrix}\right]\right\} = 0, \tag{3.14}$$

where $g_t = \mathcal{L}_g\left[T\begin{pmatrix} c_t \\ 0 \\ 0 \end{pmatrix}\right]$, $(\pi(g))_t = \sigma_{\pi(g)}\left[T\begin{pmatrix} c_t \\ 0 \\ 0 \end{pmatrix}\right]$. Equations (3.13) and (3.14) imply (3.10). □

Proposition 2. *Let* Φ *be a fundamental matrix solution of equation (3.1) with* $|\Phi| = 1$*, then* $\pi(\Phi)$ *is a fundamental matrix solution of equation (3.2).*

Proof. $\forall \alpha \in R^3$, using equation (2.3), we have

$$\mathcal{L}_\zeta(\pi(\Phi)\alpha) = \Phi\mathcal{L}_\zeta(\alpha)\Phi^{-1} \tag{3.15}$$

and

$$E\mathcal{L}_\zeta(\pi(\Phi)\alpha) = \mathcal{L}_\zeta(E\pi(\Phi)\alpha) = E\Phi\mathcal{L}_\zeta(\alpha)E\Phi^{-1} = g\Phi\mathcal{L}_\zeta(\alpha)\Phi^{-1}g^{-1} = \mathcal{L}_\zeta(\pi(g)\pi(\Phi)\alpha). \tag{3.16}$$

Hence

$$E\pi(\Phi) = \pi(g)\pi(\Phi), \tag{3.17}$$

which completes the proof of proposition 2. □

Remark 1. Let $\Phi = (\phi_{ij})_{2\times 2}$, then $\pi(\Phi)$ can be explicitly written as

$$\begin{pmatrix} \phi_{11}\phi_{22} + \phi_{12}\phi_{21} & -\zeta\phi_{11}\phi_{21} & \zeta\phi_{12}\phi_{22} \\ -2\phi_{11}\phi_{12}/\zeta & \phi_{11}^2 & -\phi_{12}^2 \\ 2\phi_{21}\phi_{22}/\zeta & -\phi_{21}^2 & \phi_{22}^2 \end{pmatrix}.$$

Using the properties of the Lie group $\pi(SL(2,R),\zeta)$ and the Lie algebra $T_e\pi(sl(2,R),\zeta)$, we have an important proposition, which plays a key role in the conserved integral.

Proposition 3. *Let* G *satisfy* $E(DG) = \pi(g)(DG)$, $(DG)_t = \sigma_\lambda(\alpha)(DG)$ *and* $\langle\cdot,\cdot\rangle$ *be the standard inner product in* R^3*, then*

$$\langle DG, C_\lambda^{-1}DG\rangle \tag{3.18}$$

is invariant along the flows of E *and* t*.*

Proof. Since $E(DG) = \pi(g)(DG)$, $(DG)_t = \sigma_\lambda(\alpha)(DG)$, and hence

$$E\langle DG, C_\lambda^{-1}DG\rangle = \langle DG, (\pi(g))^T C_\lambda^{-1}\pi(g)(DG)\rangle = \langle DG, C_\lambda^{-1}DG\rangle, \tag{3.19}$$

$$\langle DG, C_\lambda^{-1}DG\rangle_t = \langle DG, [(\sigma_\lambda(\alpha))^T C_\lambda^{-1} + C_\lambda^{-1}\sigma_\lambda(\alpha)]DG\rangle = 0. \tag{3.20}$$

The proof is complete. □

4. The Lotka–Volterra vector fields and the commutative representation

Consider the Lenard recursive equations:

$$J\xi^{(-1)} = 0, \qquad K\xi^{(j-1)} = J\xi^{(j)}. \tag{4.1}$$

Choosing a solution of the former as

$$g_n^{(-1)} = \frac{1}{2c_n}. \tag{4.2}$$

Then the latter has special polynomial solutions:

$$g_n^{(0)} = 1, \qquad g_n^{(1)} = c_{n+1} + c_n + c_{n-1}, \ldots. \tag{4.3}$$

The general solution of (4.1) is expressed as the linear combination

$$\xi^{(j)} = \alpha_0 g_n^{(j)} + \alpha_1 g_n^{(j-1)} + \cdots + \alpha_{j+1} g_n^{(-1)}, \tag{4.4}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{j+1}$ are arbitrary constants.

The so-called Lotka–Volterra vector fields are defined as $X_n^{(j)} = Jg_n^{(j)}$, and the first few members are

$$X_n^{(-1)} = 0, \qquad X_n^{(0)} = c_n(c_{n+1} - c_{n-1}), \tag{4.5$_1$}$$

$$X_n^{(1)} = c_n(c_{n+2}c_{n+1} + c_{n+1}^2 + c_{n+1}c_n - c_nc_{n-1} - c_{n-1}^2 - c_{n-1}c_{n-2}). \tag{4.5$_2$}$$

Much work on the Lotka–Volterra equation

$$c_{n,t} = c_n(c_{n+1} - c_{n-1})$$

has been carried out (see, e.g., [23–27]).

In the following, we will give the commutative representation with the 3×3 form. Let

$$G^{(N)}(\lambda, \xi) = \sum_{j=0}^{N} \xi^{(j-1)}\lambda^{N-j}. \tag{4.6}$$

We have

$$\begin{aligned}(K - \lambda J)G^{(N)} &= K\xi^{(N-1)} + \sum_{j=0}^{N-1}(K\xi^{(j-1)} - J\xi^{(j)})\lambda^{N-j} - (J\xi^{(-1)})\lambda^{N+1} \\ &= J\xi^{(N)} = \alpha_0 X^{(N)} + \alpha_1 X^{(N-1)} + \cdots + \alpha_N X^{(0)}.\end{aligned}$$

Theorem 1. *Let $G^{(N)}$ be defined in (4.6) and $\tilde{V}^{(N)} = \sigma_\lambda(DG^{(N)})$. Then*

$$(i) \quad (K - \lambda J)G^{(N)} = J\xi^{(N)}. \tag{4.7}$$

$$(ii) \quad (E\tilde{V}^{(N)})\pi(g) - \pi(g)\tilde{V}^{(N)} = \sigma_{\pi(g)}\left[T\begin{pmatrix} J\xi^{(N)} \\ 0 \\ 0 \end{pmatrix}\right]. \tag{4.8}$$

Corollary 1. *The discrete soliton equation:*

$$c_{nt} = J\xi^{(N)} \tag{4.9}$$

is equivalent to the discrete zero-curvature equation:

$$(\pi(g))_t = (E\tilde{V}^{(N)})\pi(g) - \pi(g)\tilde{V}^{(N)}. \tag{4.10}$$

5. The Poisson map and the Lie–Poisson structure

Let

$$Ey^{(j)} = \pi(g)(\lambda_j)y^{(j)}, \qquad y^{(j)} = (y^{(j1)}, y^{(j2)}, y^{(j3)})^T, \quad j = 1, \ldots, N, \tag{5.1}$$

where $\lambda_1, \ldots, \lambda_N$ are N mutual distinct real numbers. A direct calculation gives the following lemma.

Lemma 1. *Suppose $y^{(j)}$ satisfy equation (5.1). Then*

$$E(D_j\Gamma^{(j)}) - \pi(g)(\lambda_j)(D_j\Gamma^{(j)}) = T\begin{pmatrix}(K-\lambda_j J)\Gamma^{(j)}\\ 0\\ 0\end{pmatrix} = 0, \tag{5.2}$$

where $\Gamma^{(j)} = \frac{2}{\lambda_j c}y^{(j1)}$, $D_j = D(\lambda_j)$.

Using equations (3.7) and (5.2), we can easily obtain the following results, which play an important part in the Lax representation and the conserved integrals.

Proposition 4. *Let $y^{(j)}$ satisfy equation (5.1). Then*

$$\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}[E\sigma(D\Gamma^{(j)})\pi(g) - \pi(g)\sigma(D\Gamma^{(j)})] = -\sigma_{\pi(g)}\left[T\begin{pmatrix}J\sum_{j=1}^{N}\Gamma^{(j)}\\ 0\\ 0\end{pmatrix}\right]. \tag{5.3}$$

Proposition 5. *Let $G_\lambda = g^{(-1)} + \sum_{j=1}^{N}[1/(\lambda-\lambda_j)]\Gamma^{(j)}$, then*

$$(E\sigma(DG_\lambda))\pi(g) - \pi(g)\sigma(DG_\lambda) = \sigma_{\pi(g)}\left[T\begin{pmatrix}J(g^{(0)} - \sum_{j=1}^{N}\Gamma^{(j)})\\ 0\\ 0\end{pmatrix}\right]. \tag{5.4}$$

Theorem 2. $(E\sigma(DG_\lambda))\pi(g) - \pi(g)\sigma(DG_\lambda) = 0$ *is equivalent to*

$$c_n = 2\sum_{j=1}^{N}\frac{y^{(j1)}}{\lambda_j} + \frac{1}{2}\beta, \tag{5.5}$$

where β is a constant.

Substituting (5.5) into (5.1), we obtain the discrete nonlinearized eigenvalue problem

$$EY = \varphi(Y), \qquad Y = (y^{(11)}, y^{(12)}, y^{(13)}, \ldots, y^{(N1)}, y^{(N2)}, y^{(N3)})^T. \tag{5.6}$$

From equation (3.7), we know that

$$E(DG_\lambda) = \pi(g)(DG_\lambda)$$

is equivalent to

$$E\sigma(DG_\lambda)\pi(g) - \pi(g)\sigma(DG_\lambda) = 0.$$

Hence, if equation (5.5) holds, then theorem 2 and proposition 3 indicate that

$$\mathcal{F}_\lambda = \tfrac{1}{4}\langle DG_\lambda, C_\lambda^{-1}DG_\lambda\rangle \tag{5.7}$$

with $DG_\lambda = (-\frac{1}{2}\lambda - 2\sum_{j=1}^{N}y^{(j1)}/\lambda_j, 1, -\frac{1}{2}\beta - 2\sum_{j=1}^{N}y^{(j1)}/\lambda_j)^T - 2\sum_{j=1}^{N}y^{(j)}/(\lambda-\lambda_j)$ is the conserved integral of the E flow. By direct calculation, we have

$$\begin{aligned}\mathcal{F}_\lambda &= \frac{\lambda}{4}\left(\frac{1}{2} + 2\sum_{j=1}^{N}\frac{y^{(j1)}}{(\lambda-\lambda_j)\lambda_j}\right)^2\\ &\quad - \frac{1}{4}\left(\frac{1}{2}\beta + 2\sum_{j=1}^{N}\frac{y^{(j1)}}{\lambda_j} + 2\sum_{j=1}^{N}\frac{y^{(j3)}}{\lambda-\lambda_j}\right)\left(1 - 2\sum_{j=1}^{N}\frac{y^{(j2)}}{\lambda-\lambda_j}\right)\\ &= \frac{\lambda}{16} - \frac{\beta}{8} + \sum_{j=1}^{N}\frac{I^{(j)}}{\lambda-\lambda_j} + \sum_{j=1}^{N}\frac{h^{(j)}}{(\lambda-\lambda_j)^2} = \frac{\lambda}{16} - \frac{\beta}{8} + \sum_{j=1}^{\infty}\frac{F_m}{\lambda^{m+1}},\end{aligned} \tag{5.8}$$

where

$$h^{(j)} = y^{(j2)}y^{(j3)} + \frac{(y^{(j1)})^2}{\lambda_j}, \qquad j = 1, \dots, N, \tag{5.9$_1$}$$

$$I^{(j)} = \frac{1}{2}y^{(j1)} - \frac{1}{2}y^{(j3)} + \frac{\beta}{4}y^{(j2)} + \sum_{k=1}^{N}\frac{y^{(k1)}}{\lambda_k}y^{(j2)} + \sum_{k=1}^{N}\frac{y^{(k1)}}{\lambda_k}\frac{y^{(j1)}}{\lambda_j} + \sum_{k\neq j}\frac{y^{(j2)}y^{(k3)} + y^{(j3)}y^{(k2)} + (\lambda_k{}^{-1} + \lambda_j{}^{-1})y^{(j1)}y^{(k1)}}{\lambda_j - \lambda_k}, \qquad j = 1, \dots, N \tag{5.9$_2$}$$

and

$$F_m = \sum_{j=1}^{N}\lambda_j^m I^{(j)} + m\sum_{j=1}^{N}\lambda_j^{m-1}h^{(j)}, \qquad m = 0, 1, \dots, N-1. \tag{5.10}$$

We have already found the map (5.6) and its conserved integral. Now we turn to constructing the Poisson structure on R^{3N}.

Consider the Lie algebras

$$\mathcal{LA}(\lambda_j) = \{M : M^T C_{\lambda_j} + C_{\lambda_j}M = 0, M \in sl(3, R)\}. \tag{5.11}$$

Choosing $\varepsilon_1^j, \varepsilon_2^j, \varepsilon_3^j$ as a base of $\mathcal{LA}(\lambda_j)$ by

$$\varepsilon_1^j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_j & 0 \\ 0 & 0 & -\lambda_j \end{pmatrix}, \qquad \varepsilon_2^j = \begin{pmatrix} 0 & 0 & 2 \\ -\lambda_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \varepsilon_3^j = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ \lambda_j & 0 & 0 \end{pmatrix} \tag{5.12}$$

we have the commutation relations

$$[\varepsilon_1^j, \varepsilon_2^j] = \lambda_j\varepsilon_2^j, \qquad [\varepsilon_1^j, \varepsilon_3^j] = -\lambda_j\varepsilon_3^j, \qquad [\varepsilon_2^j, \varepsilon_3^j] = 2\varepsilon_1^j. \tag{5.13}$$

Let $\omega_1^j, \omega_2^j, \omega_3^j$ be a dual basis for $\mathcal{LA}(\lambda_j)^* \simeq R^3$ and $y^{(j)} = y^{(j1)}\omega_1^j + y^{(j2)}\omega_2^j + y^{(j3)}\omega_3^j$, $(j = 1, \dots, N)$. If $F : \mathcal{LA}(\lambda_j)^* \times \cdots \times \mathcal{LA}(\lambda_j)^* \to R$, then its gradient component is the vector

$$\nabla_j F = \frac{\partial F}{\partial y^{(j1)}}\varepsilon_1^j + \frac{\partial F}{\partial y^{(j2)}}\varepsilon_2^j + \frac{\partial F}{\partial y^{(j3)}}\varepsilon_3^j, \qquad j = 1, \dots, N. \tag{5.14}$$

Thus, according to [13–17], the Lie–Poisson structure matrix associated with the Lie algebra $\mathcal{LA}(\lambda_j)$ is

$$J_j = \begin{pmatrix} 0 & \lambda_j y^{(j2)} & -\lambda_j y^{(j3)} \\ -\lambda_j y^{(j2)} & 0 & 2y^{(j1)} \\ \lambda_j y^{(j3)} & -2y^{(j1)} & 0 \end{pmatrix}, \qquad j = 1, \dots, N \tag{5.15}$$

and the Lie–Poisson bracket on $\mathcal{LA}(\lambda_j)^* \times \cdots \times \mathcal{LA}(\lambda_j)^* \simeq R^{3N}$ is

$$\begin{aligned}\{F, G\} &= \sum_{j=1}^{N}\langle y^{(j)}; [\nabla_j F, \nabla_j G]\rangle = \sum_{j=1}^{N}\langle \nabla_j F, J_j\nabla_j G\rangle \\ &= \sum_{j=1}^{N}\left[\lambda_j y^{(j2)}\left(\frac{\partial F}{\partial y^{(j1)}}\frac{\partial G}{\partial y^{(j2)}} - \frac{\partial F}{\partial y^{(j2)}}\frac{\partial G}{\partial y^{(j1)}}\right)\right. \\ &\quad + \lambda_j y^{(j3)}\left(\frac{\partial F}{\partial y^{(j3)}}\frac{\partial G}{\partial y^{(j1)}} - \frac{\partial F}{\partial y^{(j1)}}\frac{\partial G}{\partial y^{(j3)}}\right) \\ &\quad \left. + 2y^{(j1)}\left(\frac{\partial F}{\partial y^{(j2)}}\frac{\partial G}{\partial y^{(j3)}} - \frac{\partial F}{\partial y^{(j3)}}\frac{\partial G}{\partial y^{(j2)}}\right)\right],\end{aligned} \tag{5.16}$$

where$\langle\cdot;\cdot\rangle$ is the natural pairing between $\mathcal{LA}(\lambda_j)$ and its dual $\mathcal{LA}(\lambda_j)^*$, $[\cdot,\cdot]$ is the ordinary Lie bracket on the Lie algebra $\mathcal{LA}(\lambda_j)$ itself.

Following the preparation above, we now prove that the map (5.6) is a Poisson map on the Poisson manifold $\{R^{3N}, \{\cdot,\cdot\}\}$.

Theorem 3. *(i) Let* $z^{(k)} = Ey^{(k)}, k = 1, \ldots, N$. *Then*

$$\sum_{j=1}^{N}(A_k^j)^T J_j(y^{(j)})A_k^j = J_k(z^{(k)}), \qquad k = 1,\ldots,N, \tag{5.17$_1$}$$

$$\sum_{j=1}^{N}(A_k^j)^T J_j(y^{(j)})A_s^j = 0, \qquad k \neq s \tag{5.17$_2$}$$

where

$$A_k^j = \begin{pmatrix} \partial z^{(k1)}/\partial y^{(j1)} & \partial z^{(k2)}/\partial y^{(j1)} & \partial z^{(k3)}/\partial y^{(j1)} \\ \partial z^{(k1)}/\partial y^{(j2)} & \partial z^{(k2)}/\partial y^{(j2)} & \partial z^{(k3)}/\partial y^{(j2)} \\ \partial z^{(k1)}/\partial y^{(j3)} & \partial z^{(k2)}/\partial y^{(j3)} & \partial z^{(k3)}/\partial y^{(j3)} \end{pmatrix}, \qquad k, j = 1, \ldots, N.$$

(ii) Let φ^* *be a pull-back map induced by* φ. *Then*

$$\varphi^*\{F, G\} = \{\varphi^* F, \varphi^* G\}, \tag{5.18}$$

that is, $Ey = \varphi(y)$ *is a Poisson map.*

Proof. From equation (5.5), we have

$$\frac{\partial c}{\partial y^{(j1)}} = \frac{2}{\lambda_j}, \qquad \frac{\partial c}{\partial y^{(j2)}} = \frac{\partial c}{\partial y^{(j3)}} = 0, \qquad j = 1, \ldots, N. \tag{5.19}$$

Using equation (5.19), through a tedious calculation, we obtain equation (5.17). According to the Poisson bracket (5.16) and (5.17), one can easily prove that

$$\begin{aligned}\{F(\varphi(y)), G(\varphi(y))\} &= \sum_{j=1}^{N}\langle\nabla_{y^{(j)}}F, J_j(y^{(j)})\nabla_{y^{(j)}}G\rangle \\ &= \sum_{k=1}^{N}\sum_{s,j=1}^{N}\langle\nabla_{z^{(k)}}F, (A_k^j)^T J_j(y^{(j)})A_s^j\nabla_{z^{(k)}}G\rangle \\ &= \sum_{k=1}^{N}\langle\nabla_{z^{(k)}}F, J_k(z^{(k)})\nabla_{z^{(k)}}G\rangle = \{F(z), G(z)\}(z),\end{aligned} \tag{5.20}$$

is the result we need.

Next we consider the problem of the integrability of the Poisson map (5.6). Regarding the generating function $\mathcal{F}_\lambda$ as a Hamiltonian in the Poisson manifold $\{R^{3N}, \{\cdot,\cdot\}\}$, the flow equation is

$$\frac{\partial}{\partial t_\lambda}y^{(j)} = \frac{1}{2}\left[\frac{1}{\lambda-\lambda_j}\sigma_\lambda(D(\lambda)G_\lambda) + W(\lambda)\right]y^{(j)}, \qquad j = 1, \ldots N, \tag{5.21}$$

where

$$W(\lambda) = \begin{pmatrix} 0 & -(D(\lambda)G_\lambda)^{(3)} & -(D(\lambda)G_\lambda)^{(2)} \\ 0 & (D(\lambda)G_\lambda)^{(2)} & 0 \\ 0 & 0 & -(D(\lambda)G_\lambda)^{(2)} \end{pmatrix}, \qquad D(\lambda)G_\lambda = \begin{pmatrix} (D(\lambda)G_\lambda)^{(1)} \\ (D(\lambda)G_\lambda)^{(2)} \\ (D(\lambda)G_\lambda)^{(3)} \end{pmatrix}.$$

□

Proposition 6.

(i)
$$h^{(j)} = y^{(j2)}y^{(j3)} + \frac{(y^{(j1)})^2}{\lambda_j}, \qquad j = 1, \ldots N \tag{5.22}$$

are N Casimir functions of the Poisson structure (5.16).

(ii) The equation for $D(\mu)G_\mu$ along the $\mathcal{F}_\lambda$ flow is
$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda}(D(\mu)G_\mu) = \frac{1}{2}\left[\frac{1}{\lambda-\mu}\sigma_\lambda(D(\lambda)G_\lambda) + W(\lambda)\right]D(\mu)G_\mu, \tag{5.23}$$

where
$$\frac{1}{\lambda-\mu}\sigma_\lambda(D(\lambda)G_\lambda) + W(\lambda) = \frac{1}{\lambda-\mu}\sigma_\mu(D(\lambda)G_\lambda) + \sigma_\mu\begin{pmatrix}\frac{1}{2}(D(\lambda)G_\lambda)^{(2)}\\ 0\\ 0\end{pmatrix}.$$

Proof.

(i) Since
$$J_j\nabla_j h^{(j)} = 0, \qquad j = 1, \ldots, N$$
thus $\forall F \in C^\infty(R^{3N})$
$$\{F, h^{(j)}\} = 0, \qquad j = 1, \ldots, N.$$

(ii) Equation (2.8) implies that
$$\sigma_\lambda(D(\lambda)G_\lambda)(D(\lambda)G_\lambda) = 0.$$

□

By equation (5.21) we have
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_\lambda}(D(\mu)G_\mu) &= \begin{pmatrix}-2\sum_{j=1}^N y_{t_\lambda}^{(j1)}/\lambda_j\\ 0\\ -2\sum_{j=1}^N y_{t_\lambda}^{(j1)}/\lambda_j\end{pmatrix} - 2\sum_{j=1}^N \frac{1}{\mu-\lambda_j}y_{t_\lambda}^{(j)}\\ &= \frac{1}{2}\left[\frac{1}{\lambda-\mu}\sigma_\lambda(D(\lambda)G_\lambda)(D(\mu)G_\mu - D(\lambda)G_\lambda) + W(\lambda)(D(\mu)G_\mu)\right]\\ &= \frac{1}{2}\left[\frac{1}{\lambda-\mu}\sigma_\lambda(D(\lambda)G_\lambda) + W(\lambda)\right]D(\mu)G_\mu.\end{aligned}$$

Using proposition 3 and equation (5.23), we have

Corollary 2. $\{\mathcal{F}_\lambda, \mathcal{F}_\mu\} = 0$, $\forall\, \lambda, \mu \in$ C;
$$\{I^{(j)}, I^{(k)}\} = 0, \qquad 1 \leqslant k, j \leqslant N;$$
$$\{F_m, F_n\} = 0, \qquad \forall\, m, n = 0, 1, \ldots.$$

In the following, we will give the functional independence of the conserved integrals $I^{(j)}$, $1 \leqslant j \leqslant N$.

Proposition 7. *The N 1-forms* $\mathrm{d}I^{(j)}$, $1 \leqslant j \leqslant N$, *are linearly independent.*

Proof. Suppose that there exist N constants $\alpha^{(j)}$, $1 \leqslant j \leqslant N$, satisfying
$$\alpha^{(1)}\mathrm{d}I^{(1)} + \alpha^{(2)}\mathrm{d}I^{(2)} + \cdots + \alpha^{(N)}\mathrm{d}I^{(N)} = 0, \tag{5.24}$$
then equation (5.23) implies
$$\alpha^{(1)}\nabla I^{(1)} + \alpha^{(2)}\nabla I^{(2)} + \cdots + \alpha^{(N)}\nabla I^{(N)} = 0. \tag{5.25}$$

□

Since

$$\nabla_j I^{(j)} = \begin{pmatrix} \frac{1}{2} + \frac{1}{\lambda_j} y^{(j2)} + 2y^{(j1)}/\lambda_j^2 \\ \frac{\beta}{4} + \sum_{k=1}^{N} y^{(k1)}/\lambda_k \\ -\frac{1}{2} \end{pmatrix} + \sum_{k \neq j} \frac{1}{\lambda_j - \lambda_k} \nabla_k h^{(k)}, \tag{5.26}$$

$$\nabla_l I^{(j)} = \begin{pmatrix} y^{(j2)}/\lambda_l \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\lambda_j - \lambda_l} \begin{pmatrix} (2/\lambda_l) y^{(j1)} \\ y^{(j3)} \\ y^{(j2)} \end{pmatrix}, \qquad l \neq j, \tag{5.27}$$

we have

$$-\frac{1}{2}\alpha^{(1)} + \alpha^{(1)} \sum_{k \neq 1} \frac{y^{(k2)}}{\lambda_1 - \lambda_k} + \alpha^{(2)} \frac{y^{(22)}}{\lambda_2 - \lambda_1} + \cdots + \alpha^{(N)} \frac{y^{(N2)}}{\lambda_N - \lambda_1} = 0, \tag{5.28}$$

that is

$$-\frac{1}{2}\alpha^{(1)} + (\alpha^{(1)} - \alpha^{(2)}) \frac{y^{(22)}}{\lambda_1 - \lambda_2} + \cdots + (\alpha^{(1)} - \alpha^{(N)}) \frac{y^{(N2)}}{\lambda_1 - \lambda_N} = 0. \tag{5.29}$$

Acting with the operators $\partial/\partial y^{(k2)}$, $2 \leqslant k \leqslant N$, on equation (5.29), we get that

$$\alpha^{(1)} - \alpha^{(k)} = 0, \qquad 2 \leqslant k \leqslant N. \tag{5.30}$$

Substituting equation (5.30) into (5.29), we obtain that $\alpha^{(1)} = 0$, which, together with equation (5.30), indicate that $\alpha^{(j)} = 0$, $1 \leqslant j \leqslant N$. Hence the N 1-forms $\mathrm{d}I^{(j)}$, $1 \leqslant j \leqslant N$, are linearly independent.

The equivalent relation of the Hamiltonian functions on $\{R^{3N}, \{\cdot, \cdot\}\}$ defined as $F(y) \simeq G(y)$ only and if only $F(y) = G(y) + C(y)$, and $C(y)$ is a Casimir function. By this definition and equation (5.8), we see that

$$\mathcal{F}_\lambda \simeq \mathcal{F}_\lambda^{(1)} = \sum_{j=1}^{N} \frac{I^{(j)}}{\lambda - \lambda_j}. \tag{5.31}$$

Theorem 4.

(i) $I^{(j)}$*,* $1 \leqslant j \leqslant N$ *are functionally independent.*

(ii) There exit N *independent function classes on the Poisson manifold* $\{R^{3N}, \{\cdot, \cdot\}\}$*, with representative elements* $\sum_{j=1}^{N} \lambda_j^m I^{(j)}$*,* $1 \leqslant m \leqslant N - 1$*.*

(iii) $F_m \simeq \sum_{j=1}^{N} \lambda_j^m I^{(j)}$*,* $1 \leqslant m \leqslant N - 1$*, i.e.* $F_0, \ldots, F_{N-1}$ *belong to* N *distinct equivalent classes.*

The N Casimir functions $h^{(j)}$, $1 \leqslant j \leqslant N$, are also functionally independent, because they are composed of distinct components and not simply a mixture. Thus by proposition 6 and theorem 4, we see that the Poisson map (5.6) has N independent Casimir functions $h^{(j)}$, $1 \leqslant j \leqslant N$, and N independent conserved integrals $I^{(j)}$, $1 \leqslant j \leqslant N$, which are in involution with respect to the Poisson bracket (5.16): hence it is integrable.

6. One reduction on the co-adjoint representative orbit

In order to find the reduction on the induced symplectic foliation by the Lie–Poisson structure (5.16), we first introduce the Lie group associated with the Lie algebra $\mathcal{LA}(\lambda_j)$. According to the theory of Lie groups, the one we need here is

$$\mathcal{LG}(\lambda_j) = \{r : r^T C_{\lambda_j} r = C_{\lambda_j}, r \in SL(3, R)\}. \tag{6.1}$$

Define $\sigma_{\mathcal{LA}(\lambda_j)} : R^3 \to \mathcal{LA}(\lambda_j)$ by

$$\sigma_{\mathcal{LA}(\lambda_j)}(\alpha) = C_{\lambda_j}^{-1} \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix}. \tag{6.2}$$

We then have the commutation relation

$$[\sigma_{\mathcal{LA}(\lambda_j)}(\alpha), \sigma_{\mathcal{LA}(\lambda_j)}(\beta)] = \sigma_{\mathcal{LA}(\lambda_j)}(C_{\lambda_j}^{-1}(\alpha \times \beta)), \tag{6.3}$$

and hence $\sigma_{\mathcal{LA}(\lambda_j)}$ is a Lie algebra isomorphism between R^3 with Lie bracket $[\alpha, \beta] = C_{\lambda_j}^{-1}(\alpha \times \beta)$ and the Lie algebra $\mathcal{LA}(\lambda_j)$. A similar result to equation (2.9) is

$$\sigma_{\mathcal{LA}(\lambda_j)}(r\alpha) = \mathrm{Ad}_r \sigma_{\mathcal{LA}(\lambda_j)}(\alpha), \tag{6.4}$$

that is the adjoint action of the Lie group $\mathcal{LG}(\lambda_j)$ on the Lie algebra $\mathcal{LA}(\lambda_j)$ is equivalent to the action of the Lie group $\mathcal{LG}(\lambda_j)$ on the isomorphic Lie algebra R^3.

The co-adjoint action of a group element $r \in \mathcal{LG}(\lambda_j)$ is the linear map $\mathrm{Ad}_r^* : \mathcal{LA}(\lambda_j)^* \to \mathcal{LA}(\lambda_j)^*$ on the dual space satisfying

$$\langle \mathrm{Ad}_r^*(\omega); \sigma_{\mathcal{LA}(\lambda_j)}(\alpha)\rangle = \langle \omega; \mathrm{Ad}_{r^{-1}} \sigma_{\mathcal{LA}(\lambda_j)}(\alpha)\rangle, \tag{6.5}$$

for all $\omega \in \mathcal{LA}(\lambda_j)^*$, $\alpha \in R^3$. Here $\langle ; \rangle$ is the natural pairing between $\mathcal{LA}(\lambda_j)$ and $\mathcal{LA}(\lambda_j)^*$.

Thus, using equation (6.4) the co-adjoint action Ad_r^* of r on $\mathcal{LA}(\lambda_j)^*$ has a matrix representation $\mathrm{Ad}_r^* = (r^{-1})^T = C_{\lambda_j} r C_{\lambda_j}^{-1}$ relative to the corresponding dual basis on $\mathcal{LA}(\lambda_j)^* \simeq R^3$.

Theorem 5.

(i) The symplectic leaves determined by the Casimir functions $h^{(j)}$, $(j = 1, \ldots, N)$ are the orbits of the co-adjoint representation of $\mathcal{LG}(\lambda_j)$.

(ii) The co-adjoint action of the Lie group $\mathcal{LG}(\lambda_j)$ on the $\mathcal{LA}(\lambda_j)^*$ is equivalent to the action of the Lie group $\pi(SL(2, R), \zeta_j)$ on the $\mathcal{LA}(\lambda_j)^*$.

(iii) Casimir functions $h^{(j)}$, $(j = 1, \ldots, N)$ are the invariant functions of the action of the Lie group $\pi(SL(2, R), \zeta_j)$ on R^3.

(iv) For each $r \in \mathcal{LG}(\lambda_j)$ (a constant matrix), the co-adjoint map Ad_r^* on $y^{(j)}$, $(j = 1, \ldots, N)$ is a Poisson mapping, or for each $\mathcal{R} \in \pi(SL(2, R), \zeta_j)$ (a constant matrix) on $y^{(j)}$, $(j = 1, \ldots, N)$ is a Poisson mapping.

Proof.

(i) Casimir functions $h^{(j)}$ can be rewritten as

$$h^{(j)} = \langle y^{(j)}, C_{\lambda_j}^{-1} y^{(j)}\rangle, \qquad j = 1, \ldots, N. \tag{6.6}$$

Hence, using $r^T C_{\lambda_j} r = C_{\lambda_j}$, we have

$$\langle (r^{-1})^T y^{(j)}, C_{\lambda_j}^{-1}(r^{-1})^T y^{(j)}\rangle = \langle y^{(j)}, r^{-1} C_{\lambda_j}^{-1}(r^{-1})^T y^{(j)}\rangle = \langle y^{(j)}, C_{\lambda_j}^{-1} y^{(j)}\rangle.$$

(ii) For

$$((r^{-1})^T)^T C_{\lambda_j}^{-1}(r^{-1})^T = r^{-1} C_{\lambda_j}^{-1}(r^{-1})^T = C_{\lambda_j}^{-1},$$

thus $(r^{-1})^T \in \pi(SL(2, R), \zeta_j)$.

(iii) Let $\mathcal{R} \in \pi(SL(2, R))$, then

$$\langle \mathcal{R} y^{(j)}, C_{\lambda_j}^{-1} \mathcal{R} y^{(j)}\rangle = \langle y^{(j)}, \mathcal{R}^T C_{\lambda_j}^{-1} \mathcal{R} y^{(j)}\rangle = \langle y^{(j)}, C_{\lambda_j}^{-1} y^{(j)}\rangle.$$

(iv) Only to prove it holds by $\pi(SL(2,R))$, denoting $z^{(j)} = \mathcal{R}y^{(j)}$, then

$$A^j_j = \mathcal{R}^T, \qquad A^j_k = 0, k \neq j,$$

where A^j_j is the Jacobian matrix of the transformation.

□

Similar to theorem 3, we only need to obtain the following equality:

$$(A^j_j)^T J_j(y^{(j)}) A^j_j = J_j(z^{(j)}), \qquad j = 1, \ldots, N.$$

Since

$$J_j(y^{(j)}) = \tfrac{1}{2}\sigma(y^{(j)}) C_{\lambda_j}$$

hence

$$\begin{aligned}(A^j_j)^T J_j(y^{(j)}) A^j_j &= \tfrac{1}{2}\mathcal{R}\sigma(y^{(j)}) C_{\lambda_j}\mathcal{R}^T = \tfrac{1}{2}\mathcal{R}\sigma(y^{(j)})\mathcal{R}^{-1}C_{\lambda_j}\\ &= \tfrac{1}{2}\sigma(\mathcal{R}y^{(j)})C_{\lambda_j} = J_j(z^{(j)}).\end{aligned}$$

Consider the common level set of the co-adjoint representative orbits

$$\{h^{(1)} = 0, \ldots, h^{(N)} = 0\}, \tag{6.7}$$

which leads to the foliation of R^{3N}. As the Poisson map (5.6) is restricted on it, a symplectic map on the symplectic manifold R^{2N} with the canonical Poisson bracket

$$(F, G) = \sum_{j=1}^{N}\left(\frac{\partial F}{\partial q^j}\frac{\partial G}{\partial p^j} - \frac{\partial F}{\partial p^j}\frac{\partial G}{\partial q^j}\right) \tag{6.8}$$

is obtained.

Let

$$Ey^{(j)} = \pi(g)(\lambda_j)y^{(j)}, \qquad j = 1, \ldots, N. \tag{6.9}$$

Define $\tau : R^2 \to R^3$ by

$$\tau(\psi) = \frac{1}{2}\begin{pmatrix}\lambda\psi^1\psi^2\\ -\zeta(\psi^1)^2\\ \zeta(\psi^2)^2\end{pmatrix}, \qquad \lambda = \zeta^2. \tag{6.10}$$

Taking

$$y^{(j)} = \tau(\psi^{(j)}) = \frac{1}{2}\begin{pmatrix}\lambda_j p^j q^j\\ -\zeta_j(p^j)^2\\ \zeta_j(q^j)^2\end{pmatrix}, \qquad \psi^{(j)} = \begin{pmatrix}p^j\\ q^j\end{pmatrix}, j = 1, \ldots, N, \tag{6.11}$$

which satisfy

$$h^{(j)} = y^{(j2)}y^{(j3)} + \frac{(y^{(j1)})^2}{\lambda_j} = 0, \qquad j = 1, \ldots, N. \tag{6.12}$$

By the relations

$$Ey^{(j)} - \pi(g)(\lambda_j)y^{(j)} = \tau(E\psi^{(j)} - g(\zeta_j)\psi^{(j)}), \qquad j = 1, \ldots, N \tag{6.13}$$

we get

$$E\psi^{(j)} = g(\zeta_j)\psi^{(j)} \qquad j = 1, \ldots, N. \tag{6.14}$$

Thus we have the map

$$E\Psi = S(\Psi), \qquad \Psi = (\psi^{(1)}, \ldots, \psi^{(N)})^T, \tag{6.15}$$

with

$$c_n = \langle p, q\rangle + \tfrac{1}{2}\beta, \tag{6.16}$$

where $\langle , \rangle$ is the standard inner product in R^{2N} and $\Lambda = \mathrm{diag}(\zeta_1, \ldots, \zeta_N)$. Equation (6.16) is actually the nonlinearization of the eigenvalue problem (3.1). From equation (6.10) we have the map $\mathcal{T} : R^{2N} \to R^{3N}$ by

$$\mathcal{T}(\Psi) = (\tau(\psi^{(1)}), \ldots, \tau(\psi^{(N)}))^T. \tag{6.17}$$

Proposition 8.

(i) $\{y^{(jk)}, y^{(js)}\} = (y^{(jk)}, y^{(js)})$, $j = 1, \ldots, N, k, s = 1, 2, 3.$

(ii) Let τ_* *be the tangent map. Then*

$$\tau_*[\psi^j](I\nabla_{\psi^j} F) = J_j \nabla_{y^{(j)}} F,$$

where $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(iii) $\mathrm{d}E\, p \wedge \mathrm{d}E\, q = \mathrm{d}p \wedge \mathrm{d}q, \qquad p = (p^1, \ldots, p^N)^T, q = (q^1, \ldots, q^N)^T.$

Proof.

(i) Using equations (6.8) and (6.11), direct calculation leads to them.

(ii) The matrix of the map $\tau_*[\psi^j]$ is

$$\begin{pmatrix} \frac{1}{2}\lambda_j q^j & \frac{1}{2}\lambda_j p^j \\ -\zeta_j p^j & 0 \\ 0 & \zeta_j q^j \end{pmatrix}$$

and

$$\frac{\partial F}{\partial p^j} = \frac{1}{2}\lambda_j q^j \frac{\partial F}{\partial y^{(j1)}} - \zeta_j p^j \frac{\partial F}{\partial y^{(j2)}}, \qquad \frac{\partial F}{\partial q^j} = \frac{1}{2}\lambda_j p^j \frac{\partial F}{\partial y^{(j1)}} + \zeta_j q^j \frac{\partial F}{\partial y^{(j3)}}.$$

Hence

$$\tau_*(I\nabla_{\psi^j} F) = \begin{pmatrix} 0 & -\frac{1}{2}\lambda_j\zeta_j(p^j)^2 & -\frac{1}{2}\lambda_j\zeta_j(q^j)^2 \\ \frac{1}{2}\lambda_j\zeta_j(p^j)^2 & 0 & \lambda_j p^j q^j \\ \frac{1}{2}\lambda_j\zeta_j(q^j)^2 & -\lambda_j p^j q^j & 0 \end{pmatrix} \nabla_{y^{(j)}} F = J_j \nabla_{y^{(j)}} F.$$

(iii) By equation (6.16), we have

$$\langle q, \mathrm{d}c \wedge \mathrm{d}p\rangle + \langle p, \mathrm{d}c \wedge \mathrm{d}q\rangle = 0.$$

□

Hence, using the above identity, a simple calculation gives

$$\mathrm{d}E\, p \wedge \mathrm{d}E\, q = \sum_{j=1}^{N} \mathrm{d}E\, p^j \wedge \mathrm{d}E\, q^j = \mathrm{d}p \wedge \mathrm{d}q.$$

Corollary 3.

(i) $\mathcal{T}^*\{F, G\} = (\mathcal{T}^*F, \mathcal{T}^*G)$*, that is,* $\mathcal{T}$ *is a Poisson map.*

(ii) $S^*(F, G) = (S^*F, S^*G)$*, that is,* S *is a symplectic map.*

Noting theorem 4, we have

Corollary 4.

(i) The symplectic map (6.15) has N *independent conserved integrals* $F_m = \sum_{j=1}^N \lambda_j^m I^{(j)}$, $m = 0, 1, \ldots, N$, *which are in involution with respect to the classical Poisson bracket (6.8).*

(ii) The symplectic map (6.15) is integrable.

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